LINEAR PROGRAMMING
AND STRUCTURAL DESIGN

II. Limit Design

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SUMMARY

"Limit Design" guides the structural engineer towards an economic design of structures that are made of ductile materials and have to carry specified loads. From the mathematical point of view, the problem can be reduced to one in linear programming, but even simple structures may lead to linear programming problems of considerable size. A method of solution is discussed that has been found efficient for structures of moderate complexity. A sketch of the historical development of limit analysis and design is given.

LINEAR PROGRAMMING AND STRUCTURAL DESIGN
II. Limit Design[*]

William Prager

In general terms, the problem discussed in the preceding lecture[**] can be described as follows: given the geometry of a structure and the fully plastic moments of its members, to find its carrying capacity for a given type of loading. This is known as the problem of "limit analysis." Today's lecture will be primarily concerned with the related but different problem of "limit design": given the geometry of a structure and the loads it has to carry, to choose the fully plastic moments of its members so as to minimize the weight of the structure.

Before entering on the discussion of limit design, let us review the method developed last time. Figure 1a shows a portal frame; the feet of the columns are build in, and the beam is rigidly joined to the columns. The fully plastic moment of each column is M_0 and that of the beam is $2M_0$. For the type of loading indicated in the figure, we wish to determine the maximum load intensity that the frame can carry.

The potentially dangerous cross sections of the frame are at the foot of each column, at the top of each column just below the joint with the beam, and at the points of application of the loads. These cross sections are numbered 1 through 6 in Fig. 1a,

and the bending moments at these sections will be denoted by
M_1 through M_0. These bending moments must satisfy three equa-
tions of equilibrium, which may be established kinematically
by considering the three "basic mechanisms" indicated in Figs.
1b, 1c, and 1d. The coefficients and right-hand sides of these
equations are listed in rows (a) through (c) of Table I, bend-
ing moments being considered positive when they tend to produce
tensile stresses in the fibers bordering on the interior of the
frame.

TABLE I

	M_1	M_2	M_3	M_4	M_5	M_6	r.h.s.	Pa/M_0
(b)	-1	3	-2	0	0	0	2Pa	3.00
(c)	0	0	-1	2	-1	0	2Pa	3.00
(d)	-1	0	1	0	-1	1	2Pa	2.00
M_1/M_0	-1	2.67	1	2	-1	1	—	—
(e)	0	3	-3	0	1	-1	0	—
(f)	-3	3	0	0	-2	2	6Pa	1.67
M_1/M_0	-1	1	.33	1.33	-1	1	—	—

Table I contains all the information necessary for formulating the linear programming problem that furnishes the load carrying capacity of the frame. We may, for instance, take the Equation (b) as defining our objective function and obtain two subsidiary conditions for the unknown bending moments by eliminating P from Equations (b) and (d). We thus obtain the following problem: to maximize

$$2Pa = -M_1 + 3M_2 - 2M_3, \tag{1}$$

subject to the equations

$$M_1 - 3M_2 + M_3 + 2M_4 - M_5 = 0, \tag{2}$$

$$- 3M_2 + 3M_3 - M_5 + M_6 = 0, \tag{3}$$

and the inequalities stipulating that M_4 must not exceed $2M_0$ in absolute value while the other five bending moments must not exceed M_0 in absolute value.

If we have a computer code for linear programming with bounded variables, we can at this stage put the problem on the computer and have it compute the load carrying capacity of the frame. In the present case, this would hardly be worth while because the problem is readily solved by the method developed in the last lecture. Considering in turn each of the mechanisms of Figs. 1b through 1d as a tentative collapse mechanism, we give the appropriate values to the bending moments at the plastic hinges and compute Pa/M_0 by applying the principle of virtual work; the results are indicated in the

last column of Table I. The mechanism of Fig. 1d is found to give the lowest value, $Pa/M_0 = 2$, the bending moments at the plastic hinges being $M_1 = M_5 = -M_0$ and $M_3 = M_6 = M_0$. Inserting these values into Equations (b) and (c), we solve for M_2 and M_4, the bending moments at the sections where the considered mechanism does not have plastic hinges. The row just below (d) in Table I lists the resulting dimensionless bending moments M_1/M_0. As M_2 exceeds its upper bound M_0, we must superimpose, on the considered mechanism (Fig. 1d), another mechanism that has a positive hinge at 2 and no hinge at 4, and yields zero total work of the loads. This mechanism is, of course, a linear combination of the three basic mechanisms (Figs. 1b through 1d). To within an irrelevant factor, the coefficients in this linear combination are determined by the following conditions: in the equation of equilibrium corresponding to the desired mechanism, the coefficient of M_2 must be positive, while that of M_4 and the right-hand side must vanish. A glance at Table I shows that the combination (b) — (d) satisfies these conditions. The resulting equation of equilibrium is given in row (e), and the corresponding mechanism is shown in Fig. 1e. Comparing the coefficients in the rows (d) and (e), we note that the smallest positive value λ for which a coefficient in the combination (d) + λ(e) vanishes is $\lambda = 1/3$. To avoid fractions, we consider the combination 3(d) + (e) which is given in row (f). The corresponding value of Pa/M_0 is 1.67 as indicated in the last

column of row (f). When the bending moments are computed as before, the values in the last row of Table I are obtained. Since these are all within the bounds for the respective moments, we have found the load carrying capacity of the frame to be $P = 1.67 \ M_0/a$. The corresponding collapse mechanism is shown in Fig. 1f.

At this stage it seems appropriate to discuss several questions which, I am sure, have occurred to some of you. First of all, is this systematic approach really necessary, or would the collapse mechanism be obvious to an experienced analyst? For the simple problem considered here, one would certainly consider the mechanism of Fig. 1f as a likely collapse mechanism because it does not have plastic hinges in the relatively strong beam. Computing first the load intensity necessary to activate this mechanism, and then the bending moments at the sections 2 and 4 as before, one easily verifies that this is indeed a collapse mechanism of the frame. In more complex structures, however, the initial guess regarding the collapse mechanism is likely to prove erroneous; the systematic procedure described above will then lead to the correct mechanism.

A question that forces itself on anybody familiar with linear programming concerns "degeneracy." Described in structural terms, degeneracy occurs when the mechanism obtained at a certain stage of our procedure has less than the "regular" number of plastic hinges. For the frame of Fig. 1a, for instance, we have six possible locations of plastic hinges

and, after eliminating P between the three equilibrium con-
ditions, two linear homogeneous equations for the bending mo-
ments M_1 through M_6. These allow us to compute two bending
moments, when the four others have been chosen in accordance
with the relative rotations in the four plastic hinges of a
tentative collapse mechanism. For the frame of Fig. 1a, the
mechanisms in Figs. 1d through 1f, which have four hinges,
are therefore "regular," while those in Figs. 1b and 1c, which
have only three hinges, are "degenerate."

In the theory of linear programming, degeneracy is es-
sentially treated by slightly modifying the problem so as to
avoid degeneracy in the modified problem. To show that a
similar treatment of degeneracy may be possible in limit anal-
ysis of structures, let us again consider the frame in Fig. 1a,
assuming that the tentative collapse mechanism is chosen ac-
cording to Fig. 1b. The three hinges of this mechanism allow
infinitesimal horizontal displacements of the cross sections
of the left-hand column only because they lie on the same
straight line. It is therefore natural to consider the frame
of Fig. 1a as the limiting case of frames that have slight
built-in kinks at 2 (Fig. 2). To avoid fractional values of
linear or angular displacements, the depth of the kink is
denoted by $4\epsilon a$ in Fig. 2. In Fig. 1b, the segments 1, 2 and
2, 3 of the left-hand column have rotated by the small amounts
θ and 2θ in the clockwise and counterclockwise senses, re-
spectively. For the frame of Fig. 2, the same rotations would
produce a vertical downward displacement of point 3 equal to

$12\epsilon a\theta$, the horizontal displacement of this point being zero
as in Fig. 1b. To make this vertical displacement possible,
we must insert an additional plastic hinge at either 4 or 5.
The first choice would not alter the load intensity $P = 3M_0/a$
needed to activate the mechanism, whereas the second choice
lowers it to $3M_0/[(1+3\epsilon)a]$. We therefore choose the additional
hinge at 5, setting $M_1 = M_3 = M_5 = -M_0$ and $M_2 = M_0$. Proceeding
as before, we are immediately led to the collapse mechanism
of Fig. 1f. While it seems likely that degeneracy can always
be treated in this fashion, no general proof of this seems to
have been given so far.

Finally, there is the question of whether it is realistic
to restrict the discussion to a single "type of loading." With
reference to Fig. 1a, for instance, we might assume the vertical
and horizontal loads to vary independently between given limits,
and ask whether any of the resulting states of loading over-
taxes the frame. Denoting the horizontal load by P as before,
and the vertical load by Q, we then have to consider all pos-
sible collapse mechanisms and specify that for each of them
the work dissipated in the plastic hinges should not be less
than the work of the applied loads. For the mechanism of
Fig. 1d, for instance, this condition provides the inequality

$$2M_0 \geq |Pa| , \tag{4}$$

where the absolute value has been used on the right-hand side
because negative values of P, i.e., horizontal loads directed

towards the left in Fig. 1, may be admitted. Using Pa/M_0 and
Qa/M_0 as rectangular Cartesian coordinates of the "load point"
in a "load plane," we see that (4) restricts the load point
to a strip that is centered at the origin. Each potential
collapse mechanism furnishes such a strip; the polygon common
to all strips (Fig. 3) contains the load points corresponding
to states of loading within the carrying capacity of the frame.
On the other hand, the given ranges of variation of P and Q
specify a rectangle in the load plane and are admissible only
if this rectangle is contained within the polygon of Fig. 3.

Let us now turn to the problem of limit design: given
the geometry of a structure and the loads it has to carry,
to determine the fully plastic moments of its members so as
to minimize its weight. For the treatment of this problem
it is customary to assume that the weight per unit length of
a structural member is proportional to its fully plastic mo-
ment M_0. This assumption greatly simplifies the analysis
because it leads to a linear rather than a nonlinear programming
problem. Whether the assumption is reasonable in a particular
case, depends on the range of cross—sectional shapes that are
at the disposal of the designer. If, for instance, all avail-
able cross—sections are geometrically similar, the unit weight
is proportional to $M_0^{2/3}$, as a simple dimensional analysis will
show. Actually, the standard I sections are not strictly
similar to each other, and the exponent 2/3 must be replaced
by one closer to unity. For a more detailed discussion of

this point, Heyman's [1]* book may be consulted. In the fol-
lowing, the weight per unit length of a member will be taken
as proportional to its fully plastic moment.

Figure 4a shows a beam on three supports subjected to
specified vertical loads. The unknown fully plastic moments
of the left- and right-hand spans will be denoted by M_0 and
M_0^*, respectively. Any plastic hinge at the central support
will form just to the left or right of this support depending
on whether $M_0 < M_0^*$ or $M_0^* < M_0$. Since it is not known before-
hand which of these inequalities applies, we must consider
the four potentially dangerous cross sections marked 1 to 4 in
Fig. 4a. The equilibrium conditions for the bending moments
M_1 through M_4 at these sections may be derived from the mechan-
isms shown in Figs. 4b through 4d. The first two of these
figures show the familiar "beam mechanisms," the last a "joint
mechanism:" the infinitesimal length of beam between the
sections 2 and 3 is tilted to produce finite rotations in the
hinges 2 and 3 but only infinitesimal deflections. The equi-
librium conditions for the four bending moments are

$$2M_1 - M_2 \qquad\qquad = 30, \qquad\qquad (5)$$

$$-M_3 + 3M_4 = 20, \qquad\qquad (6)$$

$$M_2 - M_3 \qquad\quad = 0 \qquad\qquad (7)$$

where the coefficients on the left-hand sides are dimensionless,

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*Numbers in square brackets refer to the Bibliography at
the end of this report.

while the right-hand sides represent foot-tons.

Considering a set of values M_1 through M_4 that satisfy (5), (6), and (7), choose

$$M_0 = \max \, (|M_1|, \, |M_2|) \tag{8}$$

and

$$M_0^* = \max \, (|M_3|, \, |M_4|). \tag{9}$$

The beam with these fully plastic moments cannot fail under loads that have the same ratio as in Fig. 4a but lesser intensities. This beam therefore represents a feasible design, but does not necessarily minimize the weight, which is proportional to

$$W = 20M_0 + 30M_0^* . \tag{10}$$

Our problem is to minimize W as given by (10), where M_0 and M_0^* are defined by (8) and (9) in terms of four bending moments that must satisfy (5) through (7). This can be formulated as a conventional problem in linear programming by observing that (8) is equivalent to

$$M_0 \pm M_1 \geq 0, \quad M_0 \pm M_2 \geq 0, \tag{11}$$

and that similar inequalities may be used to replace (9). It is worth noting, however, that even our extremely simple structural problem leads to a linear programming problem involving 6 unknowns subject to three equations and 8 inequalities.

As is to be expected, this static formulation of the

problem of mimimum weight design has its kinematic counterpart. The unknowns in this are the fully plastic moments M_0 and M_0^*; the conditions which they must satisfy are derived from the principle that for any tentative collapse mechanism the work dissipated in the plastic hinges should not be less than the work done by the loads.

For the beam in Fig. 4a all mechanisms can be considered as linear combinations of the basic mechanisms shown in Figs. 4b through 4d. A tentative collapse mechanism must have two of the four possible plastic hinges.inactive. Table II shows the equations of equilibrium corresponding to the six potential collapse mechanisms; the first three rows correspond to the mechanisms of Figs. 4b, 4c, and 4d, and the last three rows to combinations with two inactive hinges.

TABLE II

	M_1	M_2	M_3	M_4	ft.t.
(b)	2	−1	0	0	30
(c)	0	0	−1	3	20
(d)	0	1	−1	0	0
(e) = (b) + (d)	2	0	−1	0	30
(f) = (c) − (d)	0	−1	0	3	20
(g) = (b) − (c) + (d)	2	0	0	−3	10
fully plastic moment	M_0		M_0^*		

The hinge rotations listed in columns M_1 and M_2 go with the fully plastic moment M_0 and those in columns M_3 and M_4 with M_0^* as indicated at the bottom of Table II. Since the sign of the bending moment at a plastic hinge corresponds to that of the hinge rotation, the work dissipated in the hinges of, say, the left-hand span equals the product of M_0 by the sum of the absolute values of the hinge rotations of this span. The mechanisms of Table II therefore yield the following inequalities:

$$3M_0 \geq 30 , \qquad \text{(b)}$$
$$4M_0^* \geq 20 , \qquad \text{(c)}$$
$$M_0 + M_0^* \geq 0 , \qquad \text{(d)}$$
$$2M_0 + M_0^* \geq 30 , \qquad \text{(e)}$$
$$M_0 + 3M_0^* \geq 20 , \qquad \text{(f)}$$
$$2M_0 + 3M_0^* \geq 10 . \qquad \text{(g)}$$

The problem of minimizing W as given by (10) subject to the inequalities (b) through (g) is again a problem in linear programming. It is convenient to interpret these inequalities geometrically by using M_0 and M_0^* as the rectangular Cartesian coordinates of the "design point" in the "design plane" (Fig. 5). Each of the inequalities restricts the design point to a half-plane; the "region of admissible designs" that is common to these half-planes is shaded in Fig. 5. According to (10), all design points that correspond to a given structural weight lie on a straight line, e.g., the dotted line in Fig. 5. Since this line does not contain a point of the shaded region, no

design of this structural weight can support the given loads.
As the lines corresponding to various fixed values of W are
parallel to each other, the desired design of minimum weight
is obviously represented by the vertex V of the region of
admissible designs. Since each of the sides through this vertex
correspond to a collapse mechanism, the minimum weight design
admits two competing collapse mechanisms. We observe that the
number of competing collapse mechanisms equals the number of
dimensions of our "design space." When n fully plastic moments
are at the choice of the designer, the minimum weight design
must admit at least n competing collapse mechanisms, but may
have more than n in special cases.

Making à guess concerning the competing collapse mechan-
isms for the minimum weight design, one may compute bending
moments to ascertain whether this guess corresponds to a vertex
of the region of admissible designs. If, for instance, we
take (b) and (f) as competing mechanisms in our example, we
find (in ft.t.) $M_0 = 10$ and $M_0^* = 3.33$. The bending moments
at the plastic hinges in either mechanism must then equal the
appropriate fully plastic moments in absolute value and have
the same signs as the hinge rotations. We thus have (in ft.t.)
$M_1 = -M_2 = 10$ and $M_4 = 3.33$; equilibrium finally requires that
$M_3 = -10$, and this exceed the fully plastic moment M_0^* in
absolute value. Accordingly, the assumed combination of col-
lapse mechanisms does not correspond to a vertex of the region
of admissible designs. Applying the same technique to the

combination of (c) and (e), we find (in ft.t.) M_0 = 12.5 and M_0^* = 5.0. The bending moments with plastic hinges are M_1 = -12.5 and M_3 = $-M_4$ = 5.0, and equilibrium requires that M_2 = -5.0, which is less in absolute value than M_0. This shows that we now have a vertex of the region of admissible designs, but does not reveal whether we have attained the minimum weight.

The computation of bending moments that are compatible with both collapse mechanisms incidentally yields the following result. If the competing mechanisms have a plastic hinge in common, their rotations at this hinge must have the same sign.

Any possible linear combination of the inequalities (c) and (e) restricts the design point to a half-plane that is bounded by a line through V and contains the region of admissible designs. Since the line of min W is of this type, it must be possible to combine the inequalities (c) and (e) with positive factors in such a manner that the coefficients of M_0 and M_0^* in the resulting inequality coincide with those in the expression for W (Equation 10). As has just been shown, the corresponding combination of the mechanisms (c) and (e) cannot bring about any partial or complete cancellation of rotations in the common hinge 3. Accordingly, the inequality obtained as a positive linear combination of the inequalities (c) and (e) directly expresses the fact that in the corresponding combination of the mechanisms (c) and (e) the work dissipated in the plastic hinges must not be smaller than the work done by the loads. The minimum weight design therefore admits a collapse mechanism for which the sums of the absolute values

of the hinge rotations associated with each fully plastic moment have the same ratios as the coefficients of these moments in the expression for the structural weight. In our example, this mechanism is given by $5(c) + 10(e)$; it involves the hinge rotations 20 at 1, -15 at 3, and 15 at 4.

For the beam in Fig. 4 it was an easy matter to list all potential collapse mechanisms and the corresponding inequalities. For a more complex structure, however, this may be a formidable task. Once this is accomplished, the problem is readily set up as a problem in linear programming, which may be put on a computer. Unfortunately, no systematic method has as yet been discovered for reducing the labor of listing all potential collapse mechanisms. The criteria developed above, however, are useful aides in the search for the proper combination of mechanisms. To illustrate this, consider the two-storey frame of Fig. 6a. The feet of the columns are built in, and all joints are rigid. There are 11 potentially dangerous cross sections, labelled 1 through 11 in Fig. 6a. The bending moments at these sections will be denoted by M_1 through M_{11}. Three fully plastic moments are at the choice of the designer: M_0 for the columns of the first storey, M_0' for the ceiling beam of this storey, and M_0'' for the columns and the ceiling beam of the second storey. Accordingly, we must look for three competing collapse mechanisms. Five basic mechanisms are shown—Figs. 6b through 6f; the corresponding equations of equilibrium are listed in Table III.

TABLE III

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	ft.t.
(b)	0	0	0	0	0	0	0	-1	1	-1	1	60
(c)	1	0	0	2	-2	0	1	0	1	-1	0	320
(d)	0	0	-1	2	-1	0	0	0	0	0	0	140
(e)	0	-1	1	0	0	0	0	1	0	0	0	0
(f)	0	0	0	0	-1	1	0	0	0	0	-1	0
(g) = (c) - 2(f)	1	0	0	2	0	-2	1	0	1	-1	2	320
M	38.3	5.0	20.0	53.3	-53.3	-38.3	38.3	-15.0	15.0	-15.0	15.0	ft.t.
		M_O		M_O'		M_O			M_O''			

The structural weight is proportional to

$$W = 12M_0 + 8M_0' + 20M_0''$$ (12)

The competing mechanisms are likely to include (b) and (c), the first because it determines the minimum fully plastic moment of the upper storey, and the second because it requires a particularly large amount of work to be dissipated in the plastic hinges. The inequalities corresponding to these mechanisms are

$$4M_0'' \geq 60 ,$$ (13)

and

$$2M_0 + 4M_0' + 2M_0'' \geq 320 .$$ (14)

As (d) furnishes the inequality

$$4M_0' \geq 140,$$ (15)

it would not be possible to obtain the coefficients in (12) by a positive linear combination of the coefficients in (13) through (15). Thus, if the competing collapse mechanisms include (b) and (c), they cannot also include (d). In fact, the third competing collapse mechanism is likely to be a combination of (c) and (f), because this allows us to increase the coefficient of M_0 at the expense of that of M_0' and again requires a large dissipation of work. We try (g) = (c) −2(f) (see Table III), obtaining the inequality

$$4M_0 + 2M_0' + 4M_0'' = 320.$$ (16)

The coefficients in (13), (14), and (16) can be combined with the positive factors 2, 2/3, and 8/3 to yield the coefficients in (12). The mechanisms (b), (c), and (g) thus are admissible as competing collapse mechanisms. To investigate whether they yield the solution of our problem, we solve the equations obtained by using the equality sign in (13), (14), and (16). Thus,

$$M_0 = 38.3, \quad M_0' = 53.3, \quad M_0'' = 15.0 \tag{17}$$

(in ft.t.). Using these values with the appropriate signs for the bending moments at the plastic hinges of the three mechanisms, we compute the remaining bending moments from the equations of equilibrium obtaining the entries in row M of Table III. Since none of these exceeds the relevant fully plastic moment in absolute value, the design (17) has the minimum structural weight.

We conclude this introduction to limit analysis and design with a brief history of the field. A vague belief that a structure cannot fail as long as there exists a distribution of stresses that are in equilibrium with the applied loads and do nowhere exceed the strength of the material underlies the methods of structural analysis that were in use long before the advent of the theory of elasticity. For an unusually articulate application of this principle the reader is referred to a paper by Fourier [2]*.

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*The author is indebted to Dr. George Dantzig of RAND for this reference.

With the development of the mathematical theory of elasticity, this principle became discredited, and it was only in the second decade of this century that Kazinczy [3] in Hungary and Kist [4] in Holland pointed out that the ductility of structural steel enabled a continuous beam to adapt itself to loads far in excess of those under which the yield limit was first reached locally. The early theoretical and experimental work in the new field remained restricted to continuous beams (see, for instance, the book by Van Den Broek [5], who coined the term "limit design" for what is more accurately called limit analysis). Rigid frames were first investigated from this point of view by Baker [6], whose full-scale experiments contributed much to the acceptance of the method by structural engineers. The fundamental theorems, reflecting the static and kinematic points of view, were first rigorously established by Greenberg and Prager [7] for beams and frames and soon generalized to other types of structures by Prager [8]. The use of a complete set of linearly independent collapse mechanisms was introduced by Neal and Symonds [9]. For comprehensive presentations of theory and practice of limit analysis from the point of view of the structural engineer the reader is referred to the texts of Baker, Horne, and Heyman [10], Neal [11], and Heyman [1], which contain numerous further references.

The methods of limit design are more recent and less well developed than those of limit analysis. Heyman [12] outlined

the static approach, while Foulkes [13] developed the kine-
matic approach and recognized the relation to linear program-
ming. Livesley [14] proposed a method of steepest descent
towards the structure of minimum weight and coded it for the
EDSAC at Cambridge University. For a particular example,
Prager [15] investigated the influence of the linearization
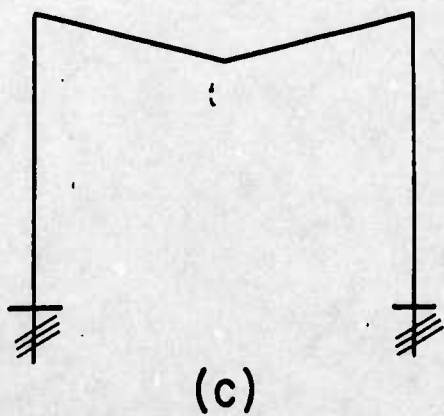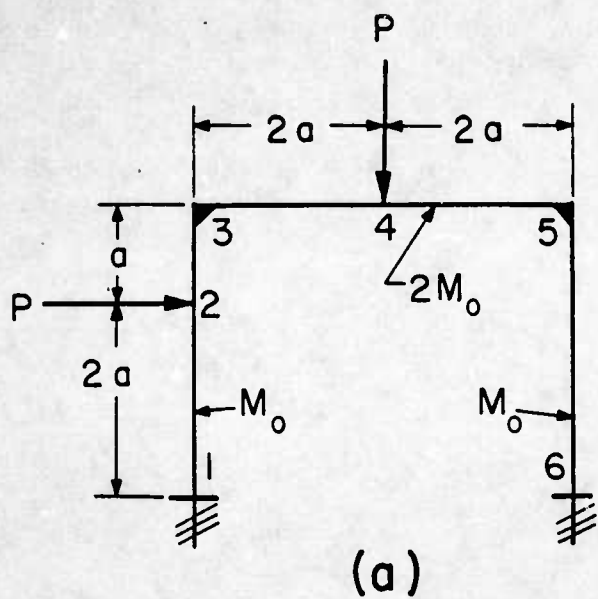of the expression for the structural weight.
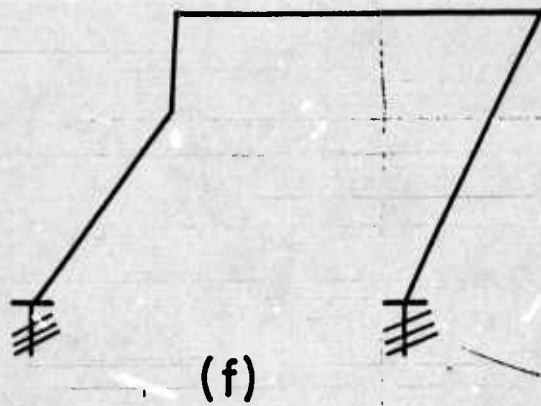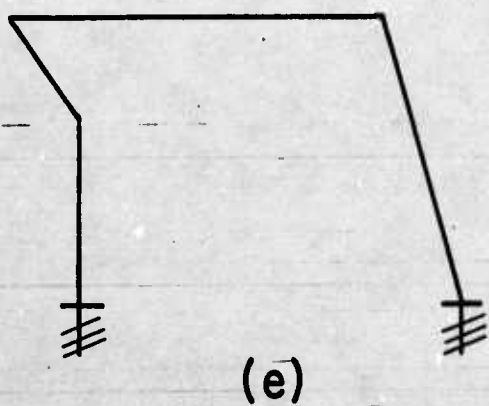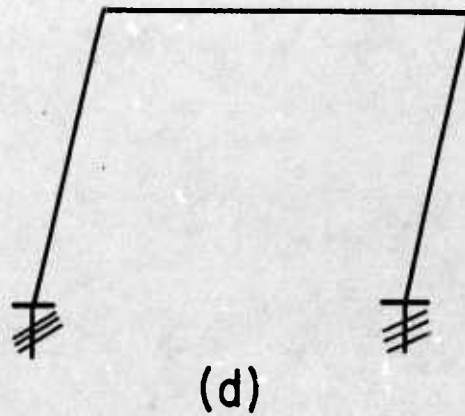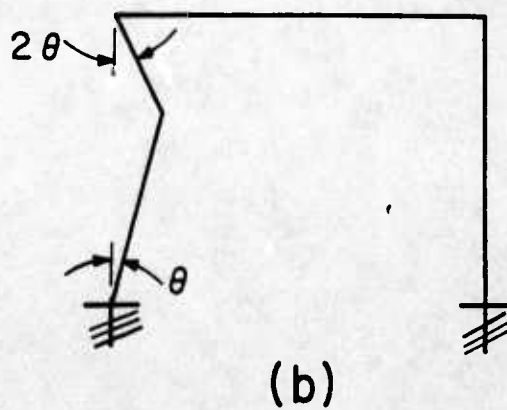
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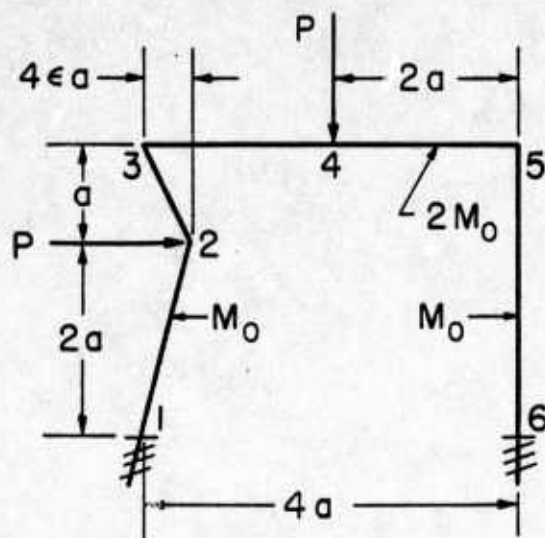
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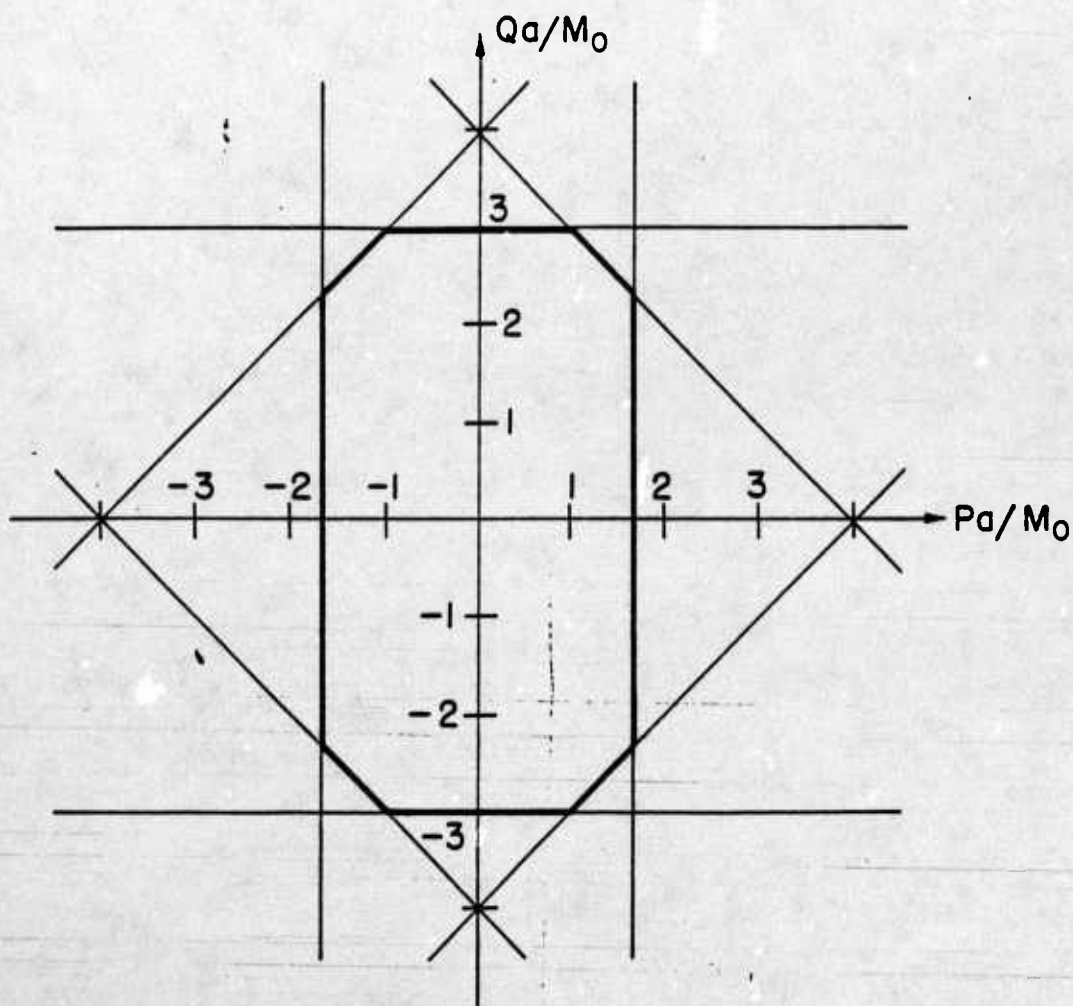
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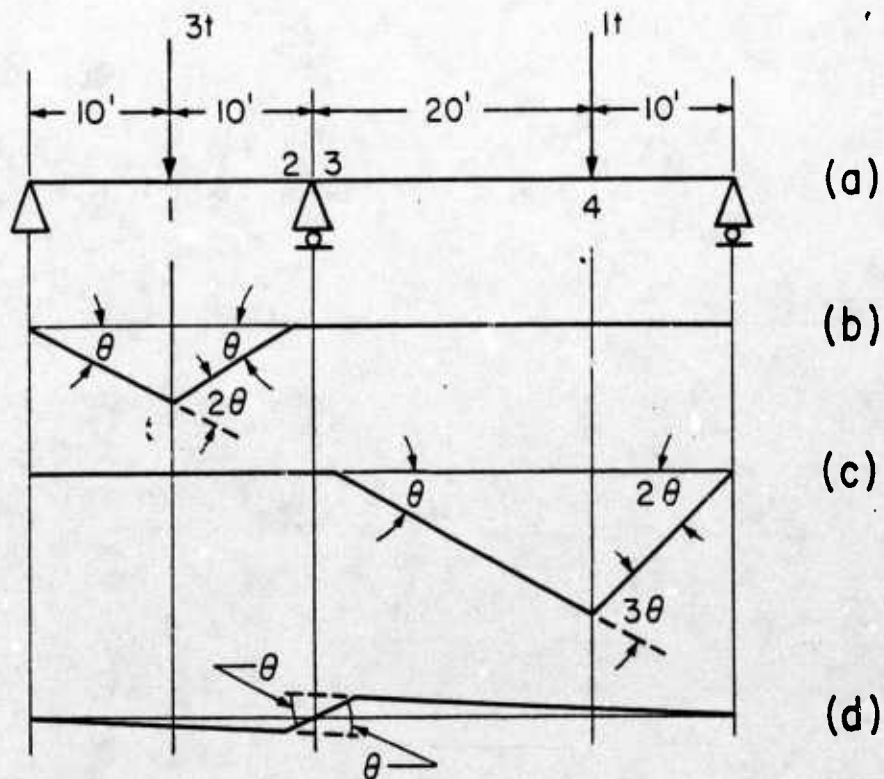
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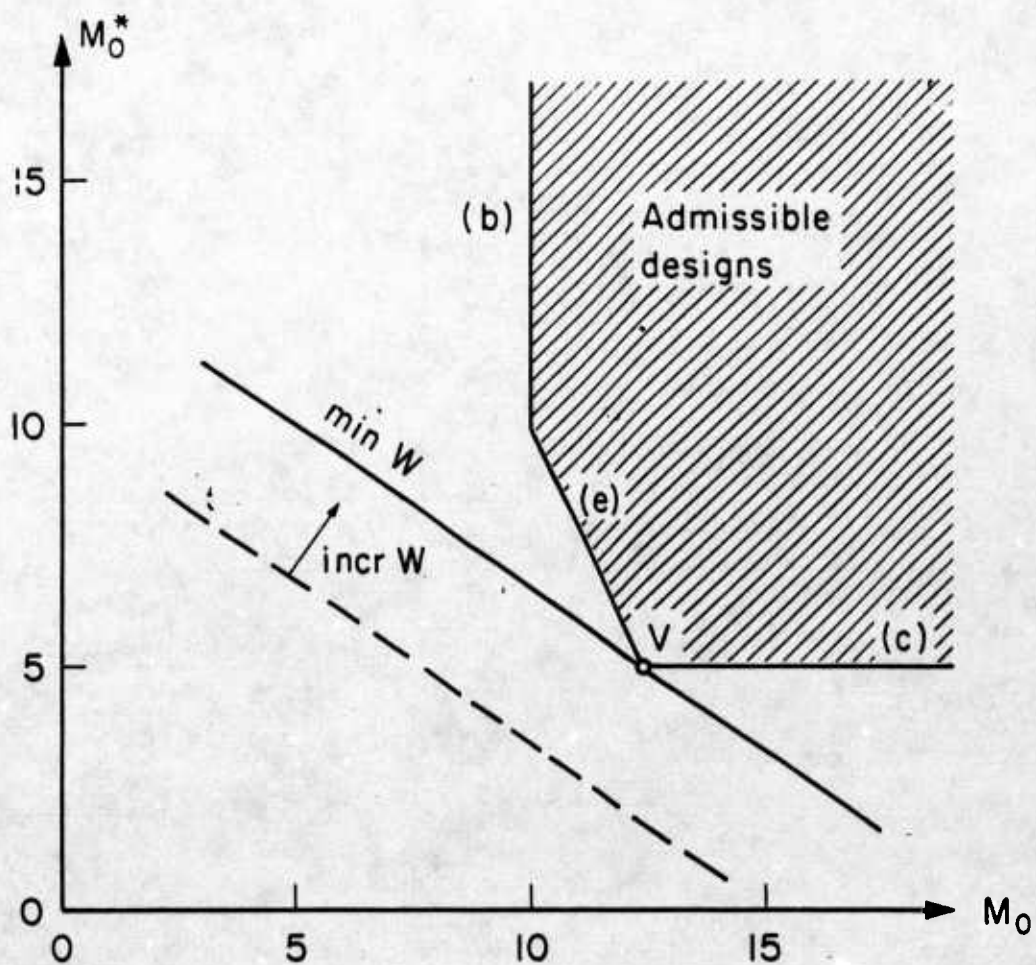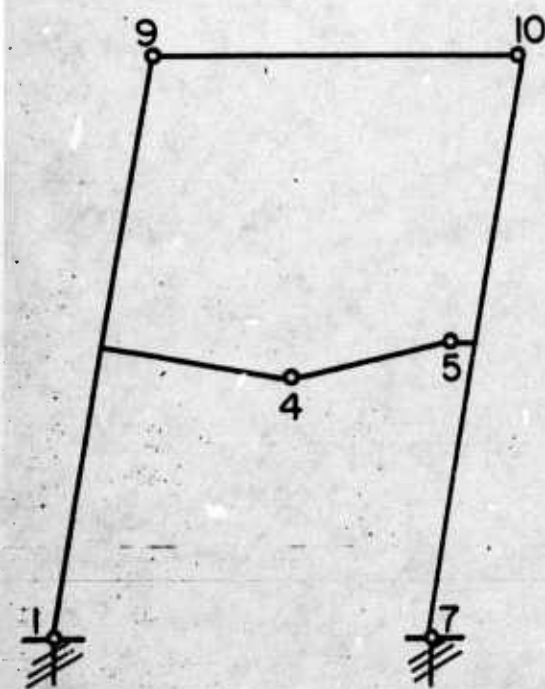
Fig. 5
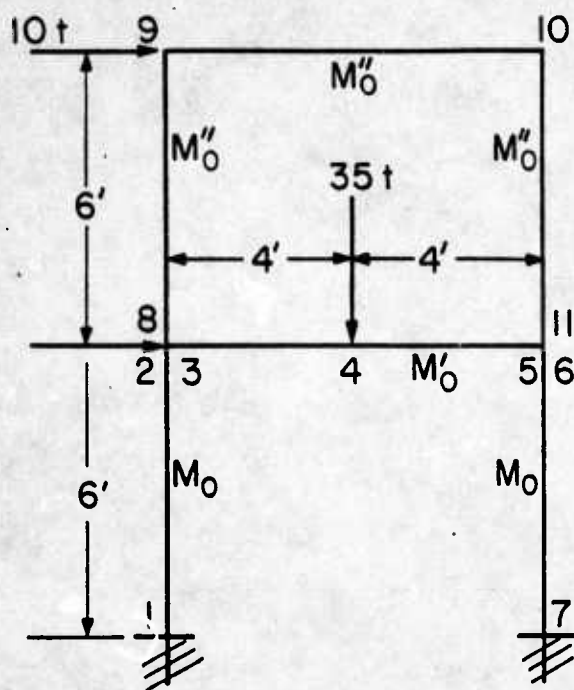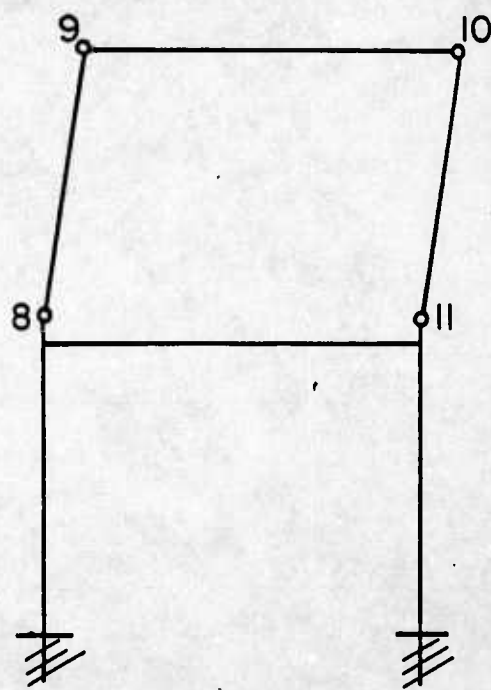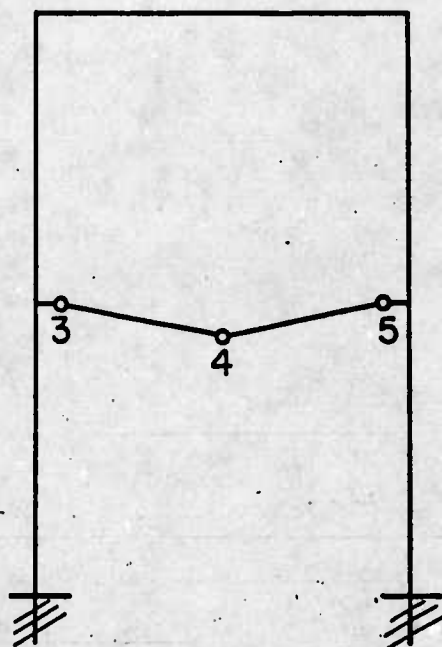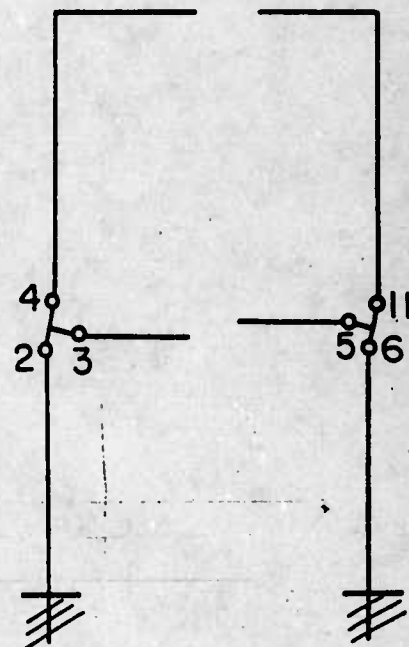
Fig. 6

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